A simple explicit bijection between (n, 2) Gog and Magog trapezoids

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A simple explicit bijection between $(n, 2)$ Gog and Magog trapezoids

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Abstract

A sub-problem of the open problem of finding an explicit bijection between alternating sign matrices and totally symmetric self-complementary plane partitions consists in finding an explicit bijection between so-called (n, k) Gog trapezoids and (n, k) Magog trapezoids. A quite involved bijection was found by Biane and Cheballah in the case $k = 2$. We give here a simpler bijection for this case.

Key words and phrases: bijection, Gog, Magog, alternating sign matrix, totally symmetric self-complementary plane partition.

AMS classification: 05A19.

1 Introduction

One of the most famous open problem in bijective combinatorics is to find an explicit bijection between alternating sign matrices of a given size and totally symmetric self-complementary plane partitions of the same size. These objects of combinatorial interest have been known since the end of the '90s to be equinumerous [And94, Zei96] but, as of today, there is no direct bijective proof of this fact. We refer the reader to [Bre99, Che11] for more information on this story.

The previous objects are in known bijections with arrays of integers called respectively *Gog* and *Magog* triangles. The triangles of size n consist in Young diagrams of shape $(n, n-1, \ldots, 2, 1)$ filled with positive integers satisfying variation conditions along vertical, horizontal and possibly oblique lines. Although they satisfy very similar variation conditions, nobody managed to find a direct bijection between these triangles. Another surprising fact is that, if we only consider the k first rows of the triangles, the objects we obtain are also equinumerous. These objects called (n, k) *trapezoids* were introduced in [MRR86] where they were conjectured to be equinumerous; this was later proved by Zeilberger [Zei96, Lemma 1].

The supposedly simplest problem of finding an explicit bijection between (n, k) Gog trapezoids and (n, k) Magog trapezoids has been solved only for $k \leq 2$. In fact, for $k = 1$, the objects are exactly the same so there is nothing to prove. There is, however, a refined conjecture by Krattenthaller [Kra] involving the number of entries equal to 1 and the number of entries equal to the maximum possible value in the first and last rows. For this conjecture, even the case $k = 1$ is nontrivial; it was proven in [Kra].

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For $k = 2$, a bijection was found by Biane & Cheballah [BC12]. Their bijection is relatively complicated and uses the so-called Schützenberger involution; it does not match the statistics of Krattenthaller's refined conjecture. In this work, we give a different bijection for this case. Our bijection is very simple and involves only one operation. It does not match Krattenthaller's statistics either.

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2 Magog and Gog trapezoids

In this work, we are solely considering $(n, 2)$ trapezoids and we furthermore impose that $n \geq 3$ in order to avoid trivialities. Let us give proper definitions (see Figure 1 for more graphical definitions and examples).

Definition 1. *Let $n \geq 3$ be an integer. An $(n, 2)$* Magog trapezoid *is an array of $2n - 1$ positive integers $m_{1,1}, \ldots, m_{1,n-1}, m_{2,1}, \ldots, m_{2,n}$ such that*

(i) $m_{i,j} \leq m_{i,j+1}$ for all $i \in \{1, 2\}$ and $j \in \{1, \ldots, n + i - 3\}$;

(ii) $m_{1,j} \leq m_{2,j} \leq j$ for all $j \in \{1, \ldots, n - 1\}$ and $m_{2,n} \leq n$.

Definition 2. *Let $n \geq 3$ be an integer. An $(n, 2)$* Gog trapezoid *is an array of $2n - 1$ positive integers $g_{1,1}, \ldots, g_{1,n}, g_{2,1}, \ldots, g_{2,n-1}$ such that*

(i) $g_{i,j} \leq g_{i,j+1}$ for all $i \in \{1, 2\}$ and $j \in \{1, \ldots, n - i\}$;

(ii) $g_{1,j} < g_{2,j} < j + 2$ for all $j \in \{1, \ldots, n - 1\}$;

(iii) $g_{1,j+1} \leq g_{2,j}$ for all $j \in \{1, \ldots, n - 1\}$.

We denote respectively by $\mathcal{M}_n$ and $\mathcal{G}_n$ the sets of $(n, 2)$ Magog and Gog trapezoids.

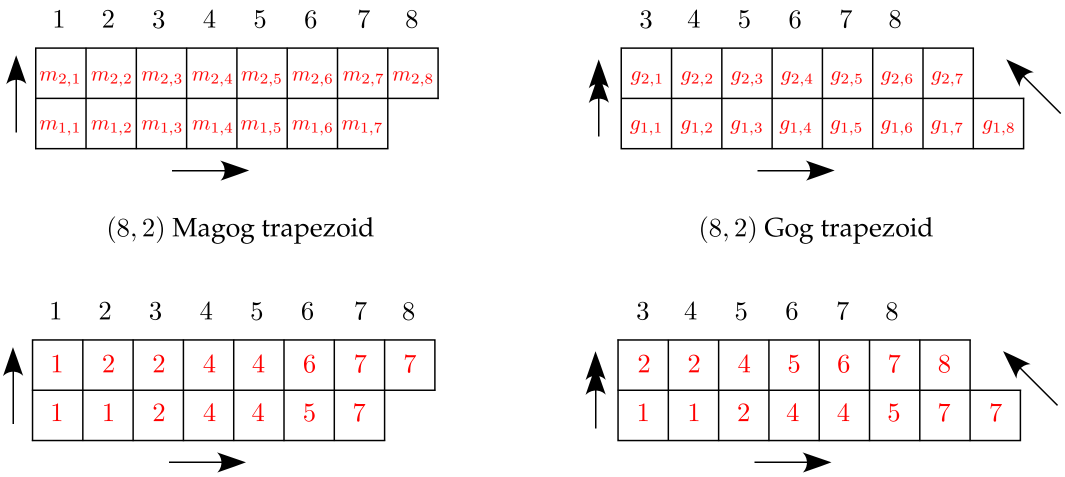


Figure 1: *The conditions satisfied by $(n, 2)$ Magog and Gog trapezoids. Every sequence formed by numbers obtained by following the direction of a simple-arrowhead (resp. a double-arrowhead) arrow is non-decreasing (resp. increasing).*

3 From Magog to Gog

Let us consider an $(n, 2)$ Magog trapezoid $M = (m_{i,j})$. We say that an integer $j \in \{1, \ldots, n-2\}$ is a *bug* if $m_{1,j+1} > m_{2,j} + 1$. For instance, 3 is the only bug of the Magog trapezoid of Figure 1. We set $\Phi_n(M) := (g_{ij})$, where (g_{ij}) is constructed as follows (see Figure 2).

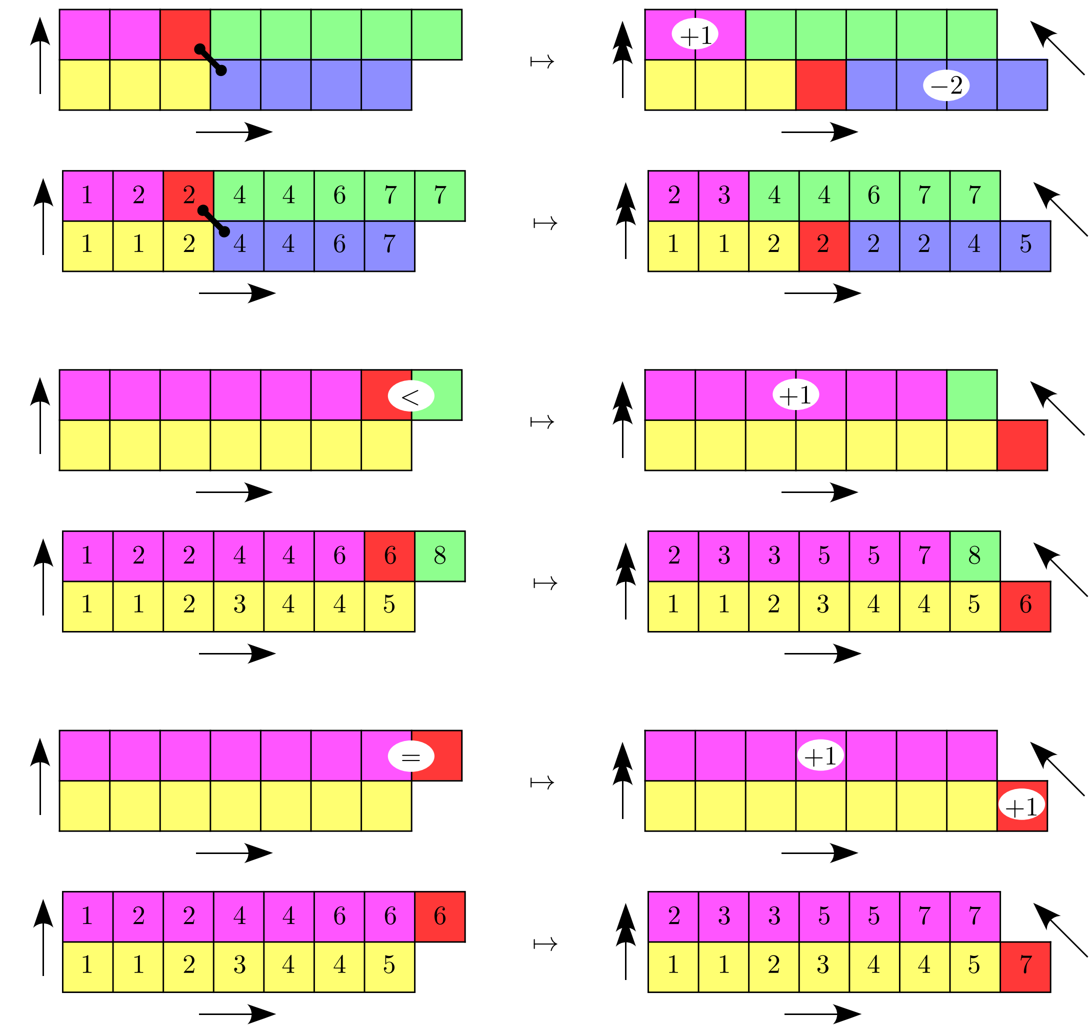


Figure 2: *The three cases of the bijection, from a Magog trapezoid to a Gog trapezoid. On the top line, the first bug is 3: it is symbolized by a small black oblique line. The colored blocks are moved and, whenever there is a tag on a block, it is added to all the elements of the block.*

First case: M has at least a bug. In this case, we let k be the smallest bug of M and we set

$g_{1,j} := m_{1,j}$ for $1 \le j \le k$; $\quad g_{1,k+1} := m_{2,k}$; $\quad g_{1,j} := m_{1,j-1} - 2$ for $k+2 \le j \le n$;

$g_{2,j} := m_{2,j} + 1$ for $1 \le j \le k-1$; $\quad g_{2,j} := m_{2,j+1}$ for $k \le j \le n-1$.

Second case: M **does not have bugs and** $m_{2,n-1} < m_{2,n}$ **.** In this case, we set

$$g_{1,j} := m_{1,j} \text{ for } 1 \le j \le n-1 ; \qquad g_{1,n} := m_{2,n-1} ;$$
$$g_{2,j} := m_{2,j} + 1 \text{ for } 1 \le j \le n-2 ; \qquad g_{2,n-1} := m_{2,n} .$$

Third case: M **does not have bugs and** $m_{2,n-1} = m_{2,n}$ **.** In this case, we set

$$g_{1,j} := m_{1,j} \text{ for } 1 \le j \le n-1 ; \qquad g_{1,n} := m_{2,n} + 1 ;$$
$$g_{2,j} := m_{2,j} + 1 \text{ for } 1 \le j \le n-1 .$$

Let us check that $\Phi_n(M) \in \mathcal{G}_n$. First, observe that, if j is not a bug, then by definition, $m_{1,j+1} \le m_{2,j} + 1$, so that the yellow and purple blocks always satisfy the oblique inequalities after the mapping. It is straightforward to verify that the other inequalities are satisfied in the second and third case. In the first case, notice that $g_{1,k} = m_{1,k} \le m_{2,k} = g_{1,k+1}$ and $g_{1,k+1} = m_{2,k} \le m_{1,k+1} - 2 = g_{1,k+2}$ as k is a bug. Furthermore, $g_{2,k-1} = m_{2,k-1} + 1 \le m_{2,k} + 1 \le m_{1,k+1} - 1 \le m_{2,k+1} - 1 = g_{2,k} - 1$ so that the horizontal inequalities are satisfied. Moreover, $g_{1,k} = m_{1,k} \le m_{2,k} \le m_{1,k+1} - 2 \le m_{2,k+1} - 2 = g_{2,k} - 2$, $g_{1,k+1} = m_{2,k} \le m_{2,k+2} - 2 = g_{2,k+1} - 2$, $g_{1,j} = m_{1,j-1} - 2 \le m_{2,j+1} - 2 = g_{2,j} - 2$ for $k+2 \le j \le n-1$, and $g_{2,j} = m_{2,j+1} < j+2$ for $k \le j \le n-1$, so that the vertical inequalities are also satisfied. Finally, the oblique inequalities are satisfied since $g_{1,k+1} = m_{2,k} \le m_{2,k+1} = g_{2,k}$ and $g_{1,j} = m_{1,j-1} - 2 \le m_{2,j} - 2 = g_{2,j-1} - 2$ for $k+2 \le j \le n$.

4 From Gog to Magog

We now consider an $(n,2)$ Gog trapezoid $G = (g_{i,j})$ and construct $\Psi_n(G) = (m_{ij})$ as follows. We define

$$k := \max \big\{ j \in \{2, \dots, n-1\} \, : g_{2,j-1} \le g_{1,j+1} + 1 \big\}. \tag{1}$$

This number is well defined as $g_{2,1} = 2 \le g_{1,3} + 1$.

First case: $k \le n-2$**.** We set

$$m_{1,j} := g_{1,j} \text{ for } 1 \le j \le k ; \qquad m_{1,j} := g_{1,j+1} + 2 \text{ for } k+1 \le j \le n-1 ;$$
$$m_{2,j} := g_{2,j} - 1 \text{ for } 1 \le j \le k-1 ; \qquad m_{2,k} := g_{1,k+1} ; \qquad m_{2,j} := g_{2,j-1} \text{ for } k+1 \le j \le n .$$

Second case: $k = n-1$ **and** $g_{1,n} < g_{2,n-1}$**.** We set

$$m_{1,j} := g_{1,j} \text{ for } 1 \le j \le n-1 ;$$
$$m_{2,j} := g_{2,j} - 1 \text{ for } 1 \le j \le n-2 ; \qquad m_{2,n-1} := g_{1,n} ; \qquad m_{2,n} := g_{2,n-1} .$$

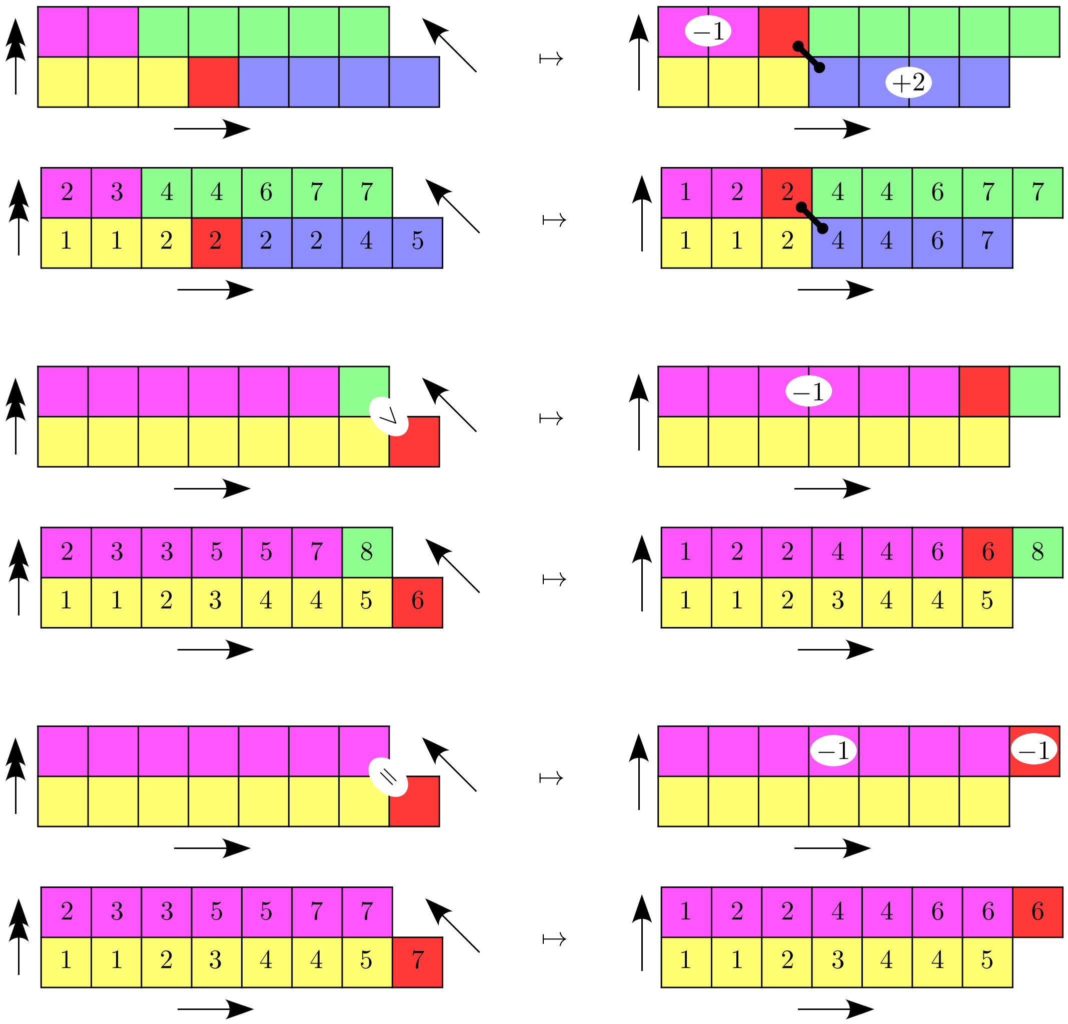


Figure 3: *The three cases of the bijection, from a Gog trapezoid to a Magog trapezoid. On the top line,* $k = 3$.

Third case: $k = n - 1$ **and** $m_{1,n} = m_{2,n-1}$**.** We set

$$m_{1,j} := g_{1,j} \text{ for } 1 \leq j \leq n-1\,;$$

$$m_{2,j} := g_{2,j} - 1 \text{ for } 1 \leq j \leq n-1 \qquad m_{2,n} := g_{1,n} - 1\,.$$

We now show that $\Psi_n(G) \in \mathcal{M}_n$. In the first and second cases, the definition of k entails that $m_{2,k-1} = g_{2,k-1} - 1 \leq g_{1,k+1} = m_{2,k}$, so that the horizontal inequalities hold. In the second case, we conclude by noticing that $m_{2,n-1} = g_{1,n} \leq g_{2,n-1} - 1 \leq n-1$ and $m_{2,n} = g_{2,n-1} \leq n$. In the first case, by definition of k, $m_{1,j} = g_{1,j+1} + 2 \leq g_{2,j-1} = m_{2,j}$ for $k+1 \leq j \leq n-1$ and, by vertical inequalities, $m_{2,j} = g_{2,j-1} \leq j$ for $k+1 \leq j \leq n$. Finally, still by definition of k, $m_{2,k} = g_{1,k+1} \leq g_{1,k+2} \leq g_{2,k} - 2 \leq k-1$. This concludes in the first case. The third case is straightforward.

5 The previous mappings are inverse one from another

We now prove that the previous mappings are bijections.

Theorem 1. *The mappings $\Phi_n : \mathcal{M}_n \to \mathcal{G}_n$ and $\Psi_n : \mathcal{G}_n \to \mathcal{M}_n$ are bijections, which are inverse one from another.*

Proof. We have already established that $\Phi_n : \mathcal{M}_n \to \mathcal{G}_n$ and $\Psi_n : \mathcal{G}_n \to \mathcal{M}_n$. It remains to show that $\Psi_n \circ \Phi_n$ and $\Phi_n \circ \Psi_n$ are the identity respectively on $\mathcal{M}_n$ and $\mathcal{G}_n$. In fact, we will see that the three cases we distinguished are in correspondence via the bijection.

First case. Let $M = (m_{i,j}) \in \mathcal{M}_n$ be a Magog trapezoid that has a bug and let k be its smallest bug. As in Section 3, we denote by $(g_{ij}) := \Phi_n(M)$. We have $g_{2,k-1} = m_{2,k-1} + 1 \le m_{2,k} + 1 = g_{1,k+1} + 1$ and, for $k+1 \le j \le n-1$, $g_{1,j+1} + 1 = m_{1,j} - 1 < m_{2,j} = g_{2,j-1}$ for $k+2 \le j \le n-1$, so that

$$\max\left\{j \in \{2,\dots,n-1\} : g_{2,j-1} \le g_{1,j+1} + 1\right\} = k.$$

As the box moving procedure of Section 4 is clearly the inverse of that of Section 3, we conclude that $\Psi_n \circ \Phi_n(M) = M$.

Let now $G = (g_{ij}) \in \mathcal{G}_n$ be such that the integer k defined by (1) is smaller than or equal to $n-2$. In order to conclude that $\Phi_n \circ \Psi_n(G) = G$, it is sufficient to show that k is the smallest bug of $(m_{i,j}) := \Psi_n(G)$. This is indeed the case as $m_{1,k+1} = g_{1,k+2} + 2 > g_{1,k+1} + 1 = m_{2,k} + 1$ and, for $1 \le j \le k-1$, $m_{1,j+1} = g_{1,j+1} \le g_{2,j} = m_{2,j} + 1$.

Second and third cases. Let $M = (m_{i,j}) \in \mathcal{M}_n$ be a bug-free Magog trapezoid and $(g_{ij}) := \Phi_n(M)$. If we are in the second case, $g_{2,n-2} = m_{2,n-2} + 1 \le m_{2,n-1} + 1 = g_{1,n} + 1$ and, if we are in the third case, $g_{2,n-2} = m_{2,n-2} + 1 \le m_{2,n} + 1 = g_{1,n}$, so that, in both cases,

$$\max\left\{j \in \{2,\dots,n-1\} : g_{2,j-1} \le g_{1,j+1} + 1\right\} = n-1$$

and we conclude as above that $\Psi_n \circ \Phi_n(M) = M$.

Let now $G = (g_{ij}) \in \mathcal{G}_n$ be such that the integer k defined by (1) is equal to $n-1$. We see that $\Phi_n \circ \Psi_n(G) = G$ by noticing that $(m_{i,j}) := \Psi_n(G)$ is bug-free as, for $1 \le j \le n-1$, $m_{1,j+1} = g_{1,j+1} \le g_{2,j} = m_{2,j} + 1$. □

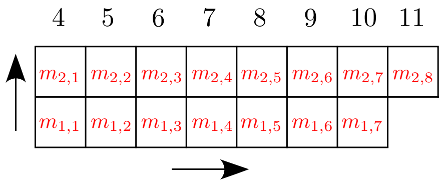
6 Extension to $(\ell, n, 2)$ trapezoids and perspectives

Our bijection can trivially be extended to $(\ell, n, 2)$ trapezoids, where $\ell \ge 0$ is an integer. An $(\ell, n, 2)$ *Magog trapezoid* is defined as an $(n, 2)$ Magog trapezoid with the difference that item *(ii)* of Definition 1 is replaced by

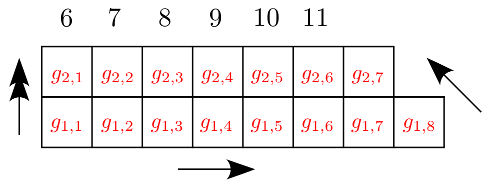
(ii') $m_{1,j} \le m_{2,j} \le j + \ell$ for all $j \in \{1,\dots,n-1\}$ and $m_{2,n} \le n + \ell$.

See Figure 4. Similarly, an $(\ell, n, 2)$ *Gog trapezoid* is defined as an $(n, 2)$ Gog trapezoid with the difference that item *(ii)* of Definition 2 is replaced by

(ii') $g_{1,j} < g_{2,j} < j + 2 + \ell$ for all $j \in \{1,\dots,n-1\}$;



$(3, 8, 2)$ Magog trapezoid



$(3, 8, 2)$ Gog trapezoid

Figure 4: *Definition of $(\ell, n, 2)$ trapezoids.*

For any $\ell \geq 1$ and $n \geq 3$, the mappings Φ_n and Ψ_n can be extended without any differences in the construction into bijections between the set of $(\ell, n, 2)$ Magog trapezoids and the set of $(\ell, n, 2)$ Gog trapezoids. The proofs can be copied almost verbatim, the only thing to do is add ℓ whenever we use one of the bounds changed by these definitions.

Unfortunately, as of today, we did not manage to extend this bijection to $(n, 3)$ trapezoids. The mapping Φ_n exchanges the sizes of two consecutive rows so that one could think that, in the case of $(n, 3)$ trapezoids, we would need to apply a similar operation several times in order to pass from a Magog to a Gog. Unfortunately, whenever a third row is present, we cannot slide the boxes of two consecutive rows without breaking the rules. This question remains under investigation.

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